Journal of the Nigerian Association of Mathematical Physics
Volume 58., (August-December 2020) pp 183-194.

NINTH ORDER BLOCK HYBRID INTEGRATORS FOR DIRECT NUMERICAL SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATIONS

[1]Olaiya O. O., [2]Joseph F. L. and [1]Modebei M. I.

[1]Department of Mathematics Programme, National Mathematical Centre, Abuja, Nigeria.
[2]Department of Mathematics, Bingham University, Karu, Abuja

Abstract

A ninth order numerical scheme is develped in this work to directly solve second order initial and boundary value problems and via the method of lines for the semi-descritization and solution for second order partial differential equations. These schemes are developed via the collocation technique and unified to form a single block of hybrid integrators. The derived method is investigated for its consistency, zero-stability and convergence and found to satisfy these characteristics. Numerical examples shows that the derived method is fond to be efficient in terms of implimentation and less computer time and in terms of accuracy when compared to existing methods in literature.

Keywords: Second order Initial-Boindary value problems, Block methods, Initial and boundary Value Problem, Linear multistep, method of lines.

1. INTRODUCTION

In this work, we consider second order partial differential equations of the form

$$y_{xx} = f(x, t, y, y_t, y_{tt}) \tag{1}$$

With appropriate initial-boundary conditions and the second order ordinanry differential equations of the form

$$y'' = f(x, y, y') \tag{2}$$

Subject to any of the boundary conditions

$$\begin{cases} y(a) = \alpha_0, y \ \ (b) = \beta_0 \\ y(a) = \alpha_0, y \ \ '(b) = \beta_1\, y_{tt}) \\ y'(a) = \alpha_1, y \ \ (b) = \beta_0 \end{cases} \tag{3}$$

Where $x \in (a, b)$, $y \in C^2[a, b]$,with $a, b, \alpha_i\beta_i \in R$ for $i = 0,1$. For the purpose of existence and uniqueness of y, we assume that the function f is a continuous function that satisfy a Lipschitz condition subject to the above initial and boundary conditions, and by extention a contraction map.

No doubt that second order Partial and ordinary differential equation finds its applications in many spheres of human endevours, these includes but not limited to particles diffusions, motions of a body, oscillations, electricity, population dynamics, finance and so on. Second order differential equations in most cases have no closed (analytic) solutions especially nonlinear ones, ence the need for numerical approximations of the solutions.

Numerical techniques are numerous and the types know no bounds .They include ;The Euler method, Runge-Kutta methods, linear Multistep method, shooting method, Finite difference method, finite element methods, e.g Galerkin method, Spectral element method. Other methods are; Spectral method base on Fourier transformation; Method of lines reduces the PDE to a large system of ordinary differential equation (ODE) Boundary Element Method (BEM) based on transforming a PDE to an integral equation on the boundary of the domains and it is popular in computational fluid dynamics, the list is endless.

Authors who have worked extensively on numerical mrethods for approximation of solution of a differential equation include but not limited to [1-7].

In this work, we consider the solution of second order differential equations by deriving linear multistep methods via the interpolation/collocation technique. These linear multistep methods are unified to form a single block which is then applied directly for solution of second order BVPs. In other to solve second order PDEs, the PDEs are semi-dicretized to form a system of second order ODEs and then the resulting systems are solve with the derived block method. For more on this approach see, [5,8].

3.2 Derivation of the Method

A 2-step block methods for the problem of the form (2) with conditions (3) is considered.
Consider the grid points given by $x_n, x_{n+1} = x_n + h, x_{n+2} = x_n + 2h$, for solving the problem in (2) on the interval $[x_n, x_{n+2}]$. We assume a trial solution $y(x)$ of (2) by a polynomial $p(x)$ given by

$$y(x) \simeq p(x) = \sum_{i=0}^{r+s-1} a_i x^i \qquad (4)$$

which on differentiating yields

$$y''(x) \simeq p''(x) = \sum_{i=2}^{r+s-1} i(i-1)a_i x^{i-2} \qquad (5)$$

with the $a_i \in \mathbb{R}$ real unknown parameters to be determined. r is the number of interpolation points and s is the number of collocation points.

2.1 Specification of the method

In this work the interval of integration considered is $[x_n, x_{n+2}]$, we thus consider the off-set points $x_{\frac{i}{4}}$, for $i = 0(1)8$. Interpolating (4) at the points x_{n+i}, for $i = \frac{1}{4}, \frac{3}{4}$, implies $r = 2$ and collocating (5) at points $x_{\frac{i}{4}}$, for $i = 0(1)\mathbf{8}$ implies $\boldsymbol{s} = \mathbf{9}$ so that (4) and (5) becomes

$$y(x) \simeq p(x) = \sum_{i=0}^{10} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10} \qquad (6)$$

which on differentiating yields

$$y''(x) \simeq p''(x) = \sum_{i=2}^{8} i(i-1)a_i x^{i-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots + 90a_{10} x^8 \qquad (7)$$

From the imposed collocation condition, the following system of algebraic equation is obtained

$$A = \begin{pmatrix}
1 & x_{\frac{1}{4}} & x_{\frac{1}{4}}^2 & x_{\frac{1}{4}}^3 & x_{\frac{1}{4}}^4 & x_{\frac{1}{4}}^5 & x_{\frac{1}{4}}^6 & x_{\frac{1}{4}}^7 & x_{\frac{1}{4}}^8 & x_{\frac{1}{4}}^9 & x_{\frac{1}{4}}^{10} \\
1 & x_{\frac{3}{4}} & x_{\frac{3}{4}}^2 & x_{\frac{3}{4}}^3 & x_{\frac{3}{4}}^4 & x_{\frac{3}{4}}^5 & x_{\frac{3}{4}}^6 & x_{\frac{3}{4}}^7 & x_{\frac{3}{4}}^8 & x_{\frac{3}{4}}^9 & x_{\frac{3}{4}}^{10} \\
0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 & 30x_0^4 & 42x_0^5 & 56x_0^6 & 72x_0^7 & 90x_0^8 \\
0 & 0 & 2 & 6x_{\frac{1}{4}} & 12x_{\frac{1}{4}}^2 & 20x_{\frac{1}{4}}^3 & 30x_{\frac{1}{4}}^4 & 42x_{\frac{1}{4}}^5 & 56x_{\frac{1}{4}}^6 & 72x_{\frac{1}{4}}^7 & 90x_{\frac{1}{4}}^8 \\
0 & 0 & 2 & 6x_{\frac{1}{2}} & 12x_{\frac{1}{2}}^2 & 20x_{\frac{1}{2}}^3 & 30x_{\frac{1}{2}}^4 & 42x_{\frac{1}{2}}^5 & 56x_{\frac{1}{2}}^6 & 72x_{\frac{1}{2}}^7 & 90x_{\frac{1}{2}}^8 \\
0 & 0 & 2 & 6x_{\frac{3}{4}} & 12x_{\frac{3}{4}}^2 & 20x_{\frac{3}{4}}^3 & 30x_{\frac{3}{4}}^4 & 42x_{\frac{3}{4}}^5 & 56x_{\frac{3}{4}}^6 & 72x_{\frac{3}{4}}^7 & 90x_{\frac{3}{4}}^8 \\
0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 & 30x_1^4 & 42x_1^5 & 56x_1^6 & 72x_1^7 & 90x_1^8 \\
0 & 0 & 2 & 6x_{\frac{5}{4}} & 12x_{\frac{5}{4}}^2 & 20x_{\frac{5}{4}}^3 & 30x_{\frac{5}{4}}^4 & 42x_{\frac{5}{4}}^5 & 56x_{\frac{5}{4}}^6 & 72x_{\frac{5}{4}}^7 & 90x_{\frac{5}{4}}^8 \\
0 & 0 & 2 & 6x_{\frac{3}{2}} & 12x_{\frac{3}{2}}^2 & 20x_{\frac{3}{2}}^3 & 30x_{\frac{3}{2}}^4 & 42x_{\frac{3}{2}}^5 & 56x_{\frac{3}{2}}^6 & 72x_{\frac{3}{2}}^7 & 90x_{\frac{3}{2}}^8 \\
0 & 0 & 2 & 6x_{\frac{7}{4}} & 12x_{\frac{7}{4}}^2 & 20x_{\frac{7}{4}}^3 & 30x_{\frac{7}{4}}^4 & 42x_{\frac{7}{4}}^5 & 56x_{\frac{7}{4}}^6 & 72x_{\frac{7}{4}}^7 & 90x_{\frac{7}{4}}^8 \\
0 & 0 & 2 & 6x_2 & 12x_2^2 & 20x_2^3 & 30x_2^4 & 42x_2^5 & 56x_2^6 & 72x_2^7 & 90x_2^8
\end{pmatrix}, \underline{x} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{pmatrix}; \qquad \underline{b} = \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{4}} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \\ f_{n+\frac{5}{4}} \\ f_{n+\frac{3}{2}} \\ f_{n+\frac{7}{4}} \\ f_{n+2} \end{pmatrix}$$

$$\underline{e} = (1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10})^T$$

Where $y_{n+k} \approx y(x_{n+k}) = y(x_n) + h,\ f_{n+k} \approx f(x_{n+k}, y_{n+k}, y'_{n+k})$. Hence, we state the following theorem without proof.

Theorem 2.1 *[8]. Let (6) and (7) be satisfied, then the 2-step continuous linear hybrid multistep method is equivalent to the equation*

$$y(x) = \underline{b}^T (A_k^{-1})^T \underline{e} \quad (8)$$

where $\underline{b}$, A and $\underline{e}$ are as defined above.

Now, invoking Theorem 2.1 and applying it, the following continuous hybrid method is derived

$$y(x) = \sum_{i=1}^{2} \alpha_{\frac{2i-1}{4}} y_{n+\frac{2i-1}{4}} + h^2 \sum_{i=0}^{2} \beta_{\frac{i}{4}} f_{n+\frac{i}{4}} \quad (9)$$

where α and β are function of t given as

$$\alpha_{\frac{1}{4}} = \frac{1}{2}(3-4t)$$

$$\alpha_{\frac{3}{4}} = \frac{1}{2}(-1+4t)$$

$$\begin{aligned}
\beta_0 &= \frac{1}{116121600} h^2(1+4t)(3+4t)(-2229-1128t+17904t^2+2688t^3-155904t^4 \\
&\quad +215040t^5+86016t^6-294912t^7+131072t^8) \\
\beta_{\frac{1}{4}} &= -\frac{1}{29030400} h^2(1+4t)(3+4t)(-23183+19508t+19600t^2+37184t^3 \\
&\quad -466688t^4+742400t^5-94208t^6-507904t^7+262144t^8) \\
\beta_{\frac{1}{2}} &= \frac{1}{29030400} h^2(1+4t)(3+4t)(317467-489028t+914992t^2-981568t^3 \\
&\quad -935168t^4+2954240t^5-1077248t^6-1490944t^7+917504t^8) \\
\beta_{\frac{3}{4}} &= -\frac{1}{29030400} h^2(1+4t)(3+4t)(-499761+246852t+1348848t^2-3349440t^3 \\
&\quad +347904t^4+5514240t^5-3280896t^6-2408448t^7+1835008t^8) \\
\beta_1 &= \frac{1}{11612160} h^2(1+4t)(3+4t)(30965+342160t-54640t^2-1533440t^3+1122560t^4 \\
&\quad +2191360t^5-2019328t^6-917504t^7+917504t^8) \\
\beta_{\frac{5}{4}} &= -\frac{1}{29030400} h^2(1+4t)(3+4t)(13199+130588t-766864t^2-1767488t^3 \\
&\quad +3194624t^4+2882560t^5-4427776t^6-1261568t^7+1835008t^8) \\
\beta_{\frac{3}{2}} &= \frac{1}{29030400} h^2(1+4t)(3+4t)(3867+37668t-221520t^2-309696t^3 \\
&\quad +1542912t^4+691200t^5-2224128t^6-344064t^7+917504t^8) \\
\beta_{\frac{7}{4}} &= -\frac{1}{29030400} h^2(1+4t)(3+4t)(817+7532t-44528t^2-48448t^3+332032t^4 \\
&\quad +35840t^5-585728t^6-16384t^7+262144t^8) \\
\beta_2 &= \frac{1}{116121600} h^2(1+4t)(3+4t)(331+2888t-17168t^2-16000t^3+130816t^4 \\
&\quad -10240t^5-241664t^6+32768t^7+131072t^8);
\end{aligned} \quad (10)$$

where $t = \frac{x-x_n}{h}$,

Evaluating (9) at the points $x = x_{n+\frac{i}{4}}$ which is equivalent to $t = \frac{i}{4}$ for $i = 0,2,4,5,6,7,8$. The following main methods are obtained

$$
\begin{aligned}
y_n = {} & \frac{3}{2}y_{n+\frac{1}{4}} - \frac{1}{2}y_{n+\frac{3}{4}} + h^2\Bigg(\frac{155171f_n}{38707200} + \frac{9421f_{n+\frac{1}{4}}}{153600} + \frac{144847f_{n+\frac{1}{2}}}{9676800} + \frac{252101f_{n+\frac{3}{4}}}{9676800} - \frac{5701f_{n+1}}{258048} \\
& + \frac{135901f_{n+\frac{5}{4}}}{9676800} - \frac{56473f_{n+\frac{3}{2}}}{9676800} + \frac{4601f_{n+\frac{7}{4}}}{3225600} - \frac{6029f_{n+2}}{38707200}\Bigg) \\
y_{n+1} = {} & -\frac{1}{2}y_{n+\frac{1}{4}} + \frac{3}{2}y_{n+\frac{3}{4}} - h^2\Bigg(\frac{743f_n}{12902400} + \frac{23183f_{n+\frac{1}{4}}}{9676800} + \frac{317467f_{n+\frac{1}{2}}}{9676800} + \frac{55529f_{n+\frac{3}{4}}}{1075200} + \frac{6193f_{n+1}}{774144} \\
& - \frac{13199f_{n+\frac{5}{4}}}{9676800} + \frac{1289f_{n+\frac{3}{2}}}{3225600} - \frac{817f_{n+\frac{7}{4}}}{9676800} + \frac{331f_{n+2}}{38707200}\Bigg) \\
y_{n+2} = {} & -\frac{5}{2}y_{n+\frac{1}{4}} + \frac{7}{2}y_{n+\frac{3}{4}} - h^2\Bigg(\frac{1453f_n}{3317760} + \frac{11147f_{n+\frac{1}{4}}}{829440} + \frac{130247f_{n+\frac{1}{2}}}{829440} + \frac{245693f_{n+\frac{3}{4}}}{829440} + \frac{79637f_{n+1}}{331776} \\
& + \frac{167717f_{n+\frac{5}{4}}}{829440} + \frac{93743f_{n+\frac{3}{2}}}{829440} + \frac{56723f_{n+\frac{7}{4}}}{829440} + \frac{12803f_{n+2}}{3317760}\Bigg) \\
y_{n+\frac{1}{2}} = {} & \frac{1}{2}y_{n+\frac{1}{4}} + \frac{1}{2}y_{n+\frac{3}{4}} + h^2\Bigg(\frac{9829f_n}{116121600} - \frac{81533f_{n+\frac{1}{4}}}{29030400} - \frac{761057f_{n+\frac{1}{2}}}{29030400} - \frac{43151f_{n+\frac{3}{4}}}{29030400} - \frac{3757f_{n+1}}{2322432} \\
& + \frac{34549f_{n+\frac{5}{4}}}{29030400} - \frac{14717f_{n+\frac{3}{2}}}{29030400} + \frac{3607f_{n+\frac{7}{4}}}{29030400} - \frac{1571f_{n+2}}{116121600}\Bigg) \\
y_{n+\frac{5}{4}} = {} & -y_{n+\frac{1}{4}} + 2y_{n+\frac{3}{4}} - h^2\Bigg(\frac{109f_n}{907200} + \frac{1103f_{n+\frac{1}{4}}}{226800} + \frac{14747f_{n+\frac{1}{2}}}{226800} + \frac{3127f_{n+\frac{3}{4}}}{28350} + \frac{1189f_{n+1}}{18144} \\
& + \frac{971f_{n+\frac{5}{4}}}{226800} + \frac{47f_{n+\frac{3}{2}}}{226800} - \frac{11f_{n+\frac{7}{4}}}{113400} + \frac{11f_{n+2}}{907200}\Bigg) \\
y_{n+\frac{3}{2}} = {} & -\frac{3}{2}y_{n+\frac{1}{4}} + \frac{5}{2}y_{n+\frac{3}{4}} - h^2\Bigg(\frac{1453f_n}{7741440} + \frac{317f_{n+\frac{1}{4}}}{43008} + \frac{187981f_{n+\frac{1}{2}}}{1935360} + \frac{326651f_{n+\frac{3}{4}}}{1935360} + \frac{33403f_{n+1}}{258048} \\
& + \frac{116287f_{n+\frac{5}{4}}}{1935360} + \frac{2557f_{n+\frac{3}{2}}}{387072} - \frac{337f_{n+\frac{7}{4}}}{645120} + \frac{331f_{n+2}}{7741440}\Bigg) \\
y_{n+\frac{7}{4}} = {} & -2y_{n+\frac{1}{4}} + 3y_{n+\frac{3}{4}} - h^2\Bigg(\frac{23f_n}{100800} + \frac{13f_{n+\frac{1}{4}}}{1350} + \frac{9847f_{n+\frac{1}{2}}}{75600} + \frac{1889f_{n+\frac{3}{4}}}{8400} + \frac{1189f_{n+1}}{6048} \\
& + \frac{4493f_{n+\frac{5}{4}}}{37800} + \frac{1649f_{n+\frac{3}{2}}}{25200} + \frac{353f_{n+\frac{7}{4}}}{75600} - \frac{29f_{n+2}}{302400}\Bigg)
\end{aligned}
\tag{11}
$$

Differentiating (10) and evaluating the derivative of (9) at the points $x = x_{n+\frac{i}{4}}$, which is equivalent to $t = \frac{i}{4}$ for $i = 0(1)8$, the following additional methods are obtained

$$
\begin{aligned}
y'_n = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}-\frac{149287hf_n}{2073600}-\frac{44629hf_{n+\frac{1}{4}}}{115200}+\frac{174373hf_{n+\frac{1}{2}}}{1451520}-\frac{2378351hf_{n+\frac{3}{4}}}{7257600}+\frac{629hf_{n+1}}{2160} \\
& -\frac{1332433hf_{n+\frac{5}{4}}}{7257600}+\frac{550087hf_{n+\frac{3}{2}}}{7257600}-\frac{8923hf_{n+\frac{7}{4}}}{483840}+\frac{29137hf_{n+2}}{14515200} \\
y'_{n+\frac{1}{4}} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}+\frac{521hf_n}{302400}-\frac{3613hf_{n+\frac{1}{4}}}{45360}-\frac{44701hf_{n+\frac{1}{2}}}{226800}+\frac{13}{225}hf_{n+\frac{3}{4}}-\frac{5039hf_{n+1}}{90720} \\
& +\frac{7523hf_{n+\frac{5}{4}}}{226800}-\frac{199hf_{n+\frac{3}{2}}}{15120}+\frac{353hf_{n+\frac{7}{4}}}{113400}-\frac{43hf_{n+2}}{129600} \\
y'_{n+\frac{1}{2}} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}-\frac{1789hf_n}{2903040}+\frac{109717hf_{n+\frac{1}{4}}}{7257600}+\frac{3047hf_{n+\frac{1}{2}}}{115200}-\frac{456943hf_{n+\frac{3}{4}}}{7257600}+\frac{1597hf_{n+1}}{45360} \\
& -\frac{9119hf_{n+\frac{5}{4}}}{483840}+\frac{51623hf_{n+\frac{3}{2}}}{7257600}-\frac{11861hf_{n+\frac{7}{4}}}{7257600}+\frac{827hf_{n+2}}{4838400} \\
y'_{n+\frac{3}{4}} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}-\frac{103hf_n}{907200}+\frac{13hf_{n+\frac{1}{4}}}{1575}+\frac{31597hf_{n+\frac{1}{2}}}{226800}+\frac{26843hf_{n+\frac{3}{4}}}{226800}-\frac{671hf_{n+1}}{30240} \\
& +\frac{971hf_{n+\frac{5}{4}}}{113400}-\frac{607hf_{n+\frac{3}{2}}}{226800}+\frac{41hf_{n+\frac{7}{4}}}{75600}-\frac{47hf_{n+2}}{907200} \\
y'_{n+1} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}-\frac{1627hf_n}{4838400}+\frac{78101hf_{n+\frac{1}{4}}}{7257600}+\frac{903097hf_{n+\frac{1}{2}}}{7257600}+\frac{40309hf_{n+\frac{3}{4}}}{161280}+\frac{1189hf_{n+1}}{9072} \\
& -\frac{150737hf_{n+\frac{5}{4}}}{7257600}+\frac{14573hf_{n+\frac{3}{2}}}{2419200}-\frac{8917hf_{n+\frac{7}{4}}}{7257600}+\frac{349hf_{n+2}}{2903040} \\
y'_{n+\frac{5}{4}} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}-\frac{149hf_n}{907200}+\frac{2039hf_{n+\frac{1}{4}}}{226800}+\frac{671hf_{n+\frac{1}{2}}}{5040}+\frac{3127hf_{n+\frac{3}{4}}}{14175}+\frac{25793hf_{n+1}}{90720} \\
& +\frac{8377hf_{n+\frac{5}{4}}}{75600}-\frac{287hf_{n+\frac{3}{2}}}{32400}+\frac{29hf_{n+\frac{7}{4}}}{22680}-\frac{31hf_{n+2}}{302400} \\
y'_{n+\frac{3}{2}} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}-\frac{5617hf_n}{14515200}+\frac{1801hf_{n+\frac{1}{4}}}{161280}+\frac{895193hf_{n+\frac{1}{2}}}{7257600}+\frac{1799953hf_{n+\frac{3}{4}}}{7257600}+\frac{3431hf_{n+1}}{15120} \\
& +\frac{302873hf_{n+\frac{5}{4}}}{1036800}+\frac{150971hf_{n+\frac{3}{2}}}{1451520}-\frac{13511hf_{n+\frac{7}{4}}}{2419200}+\frac{5809hf_{n+2}}{14515200} \\
y'_{n+\frac{7}{4}} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}+\frac{hf_n}{8640}+\frac{13hf_{n+\frac{1}{4}}}{2025}+\frac{32573hf_{n+\frac{1}{2}}}{226800}+\frac{4939hf_{n+\frac{3}{4}}}{25200}+\frac{4117hf_{n+1}}{12960} \\
& +\frac{3887hf_{n+\frac{5}{4}}}{22680}+\frac{24763hf_{n+\frac{3}{2}}}{75600}+\frac{20227hf_{n+\frac{7}{4}}}{226800}-\frac{1759hf_{n+2}}{907200} \\
y'_{n+2} = {} & -\frac{2y_{n+\frac{1}{4}}}{h}+\frac{2y_{n+\frac{3}{4}}}{h}-\frac{32273hf_n}{14515200}+\frac{203029hf_{n+\frac{1}{4}}}{7257600}+\frac{44081hf_{n+\frac{1}{2}}}{806400}+\frac{427943hf_{n+\frac{3}{4}}}{1036800}-\frac{1319hf_{n+1}}{45360} \\
& +\frac{1347173hf_{n+\frac{5}{4}}}{2419200}+\frac{74951hf_{n+\frac{3}{2}}}{7257600}+\frac{2880811hf_{n+\frac{7}{4}}}{7257600}+\frac{49613hf_{n+2}}{691200}
\end{aligned}
\tag{12}
$$

The unification of equations (11) and (12) is the required block method that must be applied to solve the required second-order problems.

3 Analysis of the Method

3.1 Order and Local Truncation Error (LTE)

The LMMs (9) is said to be of order p if

$$C_0 = C_1 = C_2 = \cdots = C_p + \mu - 1 = 0, \qquad C_{p+\mu} \neq 0.$$

Here $C_{p+\mu}$ is the error constant and

$$C_{p+\mu} h^{p+\mu} y^{(p+\mu)}(x_n)$$

is the principal Local Truncation Error (LTE) at the point x_n. The $C's$ are given by

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$$
$$C_1 = (\alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k) - (\beta_0 + \beta_1 + \cdots + \beta_k)$$
$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \frac{1}{(q-3)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-3}\beta_k), q = 2,3, \ldots$$

The LTE associated with any of (9) is given by the difference operator

$$L[y(x): h] = \sum_{i=1}^{2} \alpha_{\frac{i}{3}} y(x_n + \tfrac{i}{3}) - h^2 \sum_{j=0}^{2} \beta_{\frac{i}{3}} y''(x_n + \tfrac{i}{3}) \tag{13}$$

where $y \in C^2[a,b]$ is an arbitrary function. Expanding (13) in Taylor's series about the point x_n, the followingexpression is obtained:

$$L[y(x_n): h] = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \cdots + C_{\rho+2} h^{\rho+2} y^{\rho+2}(x_n) \tag{14}$$

\Expanding each scheme in (11) and (12), the following principal truncation errors are obtained:

$$C^0_{p+2} = \frac{-3953}{7610145177600}, C^1_{p+2} = \frac{11}{475634073600}, C^2_{p+2} = \frac{41}{72477573120}, \quad C^{\frac{1}{2}}_{p+2} = \frac{-641}{15220290355200}$$
$$C^{\frac{5}{4}}_{p+2} = \frac{11}{237817036800}, C^{\frac{3}{2}}_{p+2} = \frac{269}{3044058071040}, C^{\frac{7}{4}}_{p+2} = \frac{11}{237817036800}, \quad C'^0_{p+2} = \frac{1651777}{251134790860800}$$
$$C'^{\frac{1}{4}}_{p+2} = \frac{-7453}{7847962214400}, C'^{\frac{1}{2}}_{p+2} = \frac{112129}{251134790860800}, C'^{\frac{3}{4}}_{p+2} \frac{-589}{7847962214400}, C'^1_{p+2} \frac{9079}{35876398694400}$$
$$C'^{\frac{5}{4}}_{p+2} = \frac{-589}{7847962214400}, C'^{\frac{3}{2}}_{p+2} = \frac{112129}{251134790860800}, C'^{\frac{7}{4}}_{p+2} = \frac{-7453}{7847962214400}, C'^2_{p+2} = \frac{1651777}{251134790860800}$$

The above blocked method (11) and (12) is of uniform order $p = 9$

3.3.2 Consistency of the Methods

The LMM (9) is said to be consistent if it has order $p \geq 1$ and the first and second characteristic polynomials which are defined respectively, as

$$\rho(r) = \sum_{j=0}^{k} \alpha_j z^j \tag{15}$$

and

$$\sigma(r) = \sum_{j=0}^{k} \beta_j z^j \tag{16}$$

where r is the principal root, satisfy the following conditions:

$$\sum_{j=0}^{k} \alpha_j = 0 \tag{17}$$

$$\rho(1) = \rho'(1) = 0 \tag{18}$$

and

$$\rho''(1) = 2!\,\sigma(1) \tag{19}$$

See [1,4,9].

Consider the main method in (11) given as

$$y_{n+2} = -\frac{5}{2} y_{n+\frac{1}{4}} + \frac{7}{2} y_{n+\frac{3}{4}} - h^2 \left(\frac{1453 f_n}{3317760} - \frac{11147 f_{n+\frac{1}{4}}}{829440} - \frac{130247 f_{n+\frac{1}{2}}}{829440} - \frac{245693 f_{n+\frac{3}{4}}}{829440} - \frac{79637 f_{n+1}}{331776} - \frac{167717 f_{n+\frac{5}{4}}}{829440} - \frac{93743 f_{n+\frac{3}{2}}}{829440} - \frac{\mathbf{56723} f_{n+\frac{7}{4}}}{\mathbf{829440}} - \frac{\mathbf{12803} f_{n+2}}{\mathbf{3317760}} \right) \tag{20}$$

The condition (17) is satisfied. the first characteristic equation for (11) is given as:

$$\rho(r) = r^2 + \frac{5}{2}r^{\frac{1}{4}} - \frac{7}{2}r^{\frac{3}{4}} \quad (21)$$

$$\rho'(r) = \frac{5}{8r^{3/4}} - \frac{21}{8r^{1/4}} + 2r \quad (22)$$

Here $\rho(1) = 0$, $\rho'(1) = 0$. Therefore, (18) is satisfied. The second characteristic polynomial for (11) is given as

$$\sigma(r) = -\frac{1453}{3317760} + \frac{11147r^{\frac{1}{4}}}{829440} + \frac{130247r^{\frac{1}{2}}}{829440} + \frac{245693r^{\frac{3}{4}}}{829440} + \frac{79637r}{331776} + \frac{167717r^{\frac{5}{4}}}{829440} + \frac{93743r^{\frac{3}{2}}}{829440} + \frac{56723r^{\frac{7}{4}}}{829440} + \frac{12803r^2}{3317760} \quad (23)$$

$$\sigma(1) = \frac{35}{32} \quad (24)$$

$$\rho''(r) = 2 - \frac{15}{32r^{7/4}} + \frac{21}{32r^{5/4}} \qquad \rho''(1) = \frac{35}{16} \quad (25)$$

Hence condition (19) is satisfied. Conclusively, the hybrid method is consistent.

3.3.3 Zero Stability

To establish hat the methods are zero stable, each of the method in block form are solved simultaneously to obtain all the y_i and y'_i's for appropriate index i, see [10]. For the method (11) and its additional methods in (20), they are taken in block form and solved simultaneously to obtain $\boldsymbol{y}_{\frac{i}{4}}$ for $\boldsymbol{i = 0(1)8}$ to obtain the following block method.

A numerical method is zero-stable if the solutions remain bounded as $h \to 0$, which means that the method does not provide solutions that grow unbounded as the number of steps increases, see [10]. To show the zero-stability of the block method (11)-(12), we take $\boldsymbol{h \to 0}$ the method may be rewritten in matrix form as

Definition 3.1 *The two step hybrid block method (11)-(12) is said to be zero stabile if the number of root of the first characteristic equation* $|\rho(r)| < 1$ *and if* $|\rho(r)| = 1$, *then the multiplicity of* $\rho(r)$ *must not exceed 2.*

To show the zero-stability of the block method (11)-(12), we take $h \to 0$ the method may be rewritten in matrix form as

$$A_0 Y_n = A_1 Y_{n-1} \quad (26)$$

$Y_n = (Y_n^0, Y_n^1)^T$

$Y_n^0 = (y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, y_{n+\frac{3}{4}}, y_{n+1}, y_{n+\frac{5}{4}}, y_{n+\frac{3}{2}}, y_{n+\frac{7}{4}}, y_{n+2})^T$

$Y_n^0 = (y'_{n+\frac{1}{4}}, y'_{n+\frac{1}{2}}, y'_{n+\frac{3}{4}}, y'_{n+1}, y'_{n+\frac{5}{4}}, y'_{n+\frac{3}{2}}, y'_{n+\frac{7}{4}}, y'_{n+2})^T$

$A_0 = I_{16\times16}$ identity matrix and $A_1 = I_{16\times16}$ matrix given by

$$A_1 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}; \quad A_{11} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}; \quad A_{22} = \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&1&0&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix}$$

The characteristic polynomial of the matrix A_{11} is given as $|A_{11} - \lambda I|$, that is $\lambda^7(\lambda - 1) = 0$ with root $\lambda_j = 0$ for $j = 1, \ldots, 7$ and $\lambda_8 = 1$.

The characteristic polynomial of the matrix A_{22} is given as $|A_{22} - \lambda I|$, that is $\lambda^7(\lambda - 1) = 0$ with root $\lambda_j = 0$ for $j = 1, \ldots, 7$ and $\lambda_8 = 1$.

Hence, the method (11)-(12) is zero stable.

3.3.4 Convergence of the Methods

Definition 3.2 *(Convergence) An LMM is said to be convergent if and only if it is consistent and zero-stable.*

By the above definition, the derived hybrid methods are convergent.

3.3.6 Implementation of Method

The implementation is such that the block method is solved at once simultaneously as detailed in the below algorithm.
Block Algorithm for BHI

1 begin procedureENTER Partitions $(a, b, N, h,$ variables$)$
2 For $x_n = x_{n-1} + h,\ n = 1, \ldots, N,\ h = \frac{b-a}{N}$
3 Generate system from block
4 Solve [System, variables]
5 Obtain y_n
6 end procedure

4.0 NUMERICAL EXAMPLES

In this chapter, the performance of the developed two step hybrid block scheme is examined. the exact and approximate solution are tabulated. The tables below shows the numerical results of the new developed scheme with exact solution for solving the problem and the result of the developed scheme are more accurate than existing methods.

Example 1.

Consider the PDE.

$$\kappa \frac{\partial y^2}{\partial x^2} - \frac{\partial y}{\partial t} = 0$$
$$y(0,t) = y(1,t) = 0, \qquad y(x,0) = \sin\pi x + \sin\omega\pi x, \qquad \omega > 1 \tag{27}$$

The analytic solution is $y(x,t) = e^{-\pi^2\kappa t}\sin\pi x + e^{-\omega^2\pi^2\kappa t}\sin\pi x$

Following [11], (27) becomes

$$\frac{\mathrm{d}y_m^2}{\mathrm{d}x^2} = \frac{1}{\kappa}\frac{y(x\ ,\quad t_{m+1}) - y(x\ ,\quad t_{m-1})]}{(\Delta t)} \tag{28}$$

$$y_m(0,t_m) = y_m(1,t_m) = 0, \qquad y_m(x,0) = \sin\pi x + \sin\omega\pi x, \qquad \kappa > 1$$

where $t_m = m\Delta t, \qquad m = 0,1, \ldots, M;\ y_m(x) \approx y(x,t_m), y(x) = [y_0(x), y_1(x), \ldots, y_{M-1}(x)]^T$,
hence (28) becomes the system $\frac{d^2 y_m(x)}{dx^2} = f(x\ , y_m)$ which is in the form of (2), where $f(x,t_m) = Ay + G$ and A is an $M-1$ square matrix, G is a vector of constants.

Table 1: Exact and Numerical solution for Example 1.

x	Exact	HBI	Error
0.0	1.65341E-9	1.65341E-9	0
0.1	6.16242E-10	6.16242E-10	2.25E-14
0.2	2.29678E-10	2.29678E-10	8.12E-14
0.3	8.56029E-11	8.56029E-11	4.23E-14
0.4	3.19048E-11	3.19048E-11	7.87E-14
0.5	1.18911E-11	1.18911E-11	7.53E-14
0.6	4.43194E-12	4.43194E-12	5.83E-14
0.7	1.65181E-12	1.65181E-12	6.84E-14

0.8	6.15646E-13	6.15646E-13	3.98E-15
0.9	2.29456E-13	2.29456E-13	1.77E-15

Table 2: Comparison of maximum errors obtained in different methods for Example 1 at $t = 1$.

ω	HBI	BHSDA
1	1.011×10^{-14}	2.64×10^{-6}
2	1.075×10^{-14}	1.32×10^{-6}
3	1.098×10^{-14}	1.32×10^{-6}
5	1.032×10^{-14}	1.32×10^{-6}
10	1.012×10^{-14}	1.32×10^{-6}

BHSDA is L -Stable Block Hybrid Second Derivative Algorithm in [11].

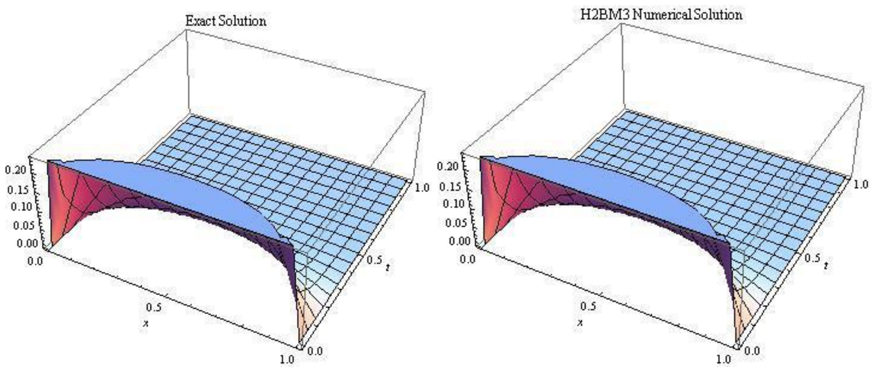

Figure 1: Surface plots for the Exact and numerical solution for Example 1

Tables 1-2 shows the comparison of exact numerical solutions and the errors for Example 1. Table 3 shows the comparison of maximum errors obtained for example 1 using the derived method and the method in [11]. This shows the superiorly of the derived method. Figure 1 show the surface plots for the exact solution and Numerical solutions for Example 1.

Example 2.

Consider the PDE in [8]:

$$\frac{\partial y^2}{\partial x^2} + \frac{\partial y^2}{\partial t^2} = -32\pi^2 \sin(4\pi x), \qquad x \in [0,1]$$
$$y(\pm 1, t) = y(x, \pm 1) = 0, \qquad t > 0 \tag{29}$$

The analytic solution is $y(x,t) = \sin(4\pi x)\sin(4\pi t)$.

Following [11], (29) becomes

$$\frac{\mathrm{d}y_m^2}{\mathrm{d}x^2} = -\frac{y(x\ ,\quad t_{m+1}) - 2y(x\ ,\quad t_m) + y(x\ ,\quad t_{m-1})]}{(2\Delta t)} - 32\pi^2 \sin(4\pi x) \tag{30}$$
$$y_m(\pm 1, t_m) = y_m(\mathrm{x}, \pm 1) = 0,$$

where $t_m = m\Delta t, \qquad m = 0,1,\dots,M;\ y_m(x) \approx y(x, t_m), y(x) = [y_0(x), y_1(x), \dots, \mathrm{y}_{M-1}(x)]^T$, hence (30) becomes the system $\frac{d^2 y_m(x)}{dx^2} = f(x\ \ , y_m)$ which is in the form of (2). where $f(x, t_m) = Ay + G$ and A is an $M-1$ square matrix, G is a vector of constants.

Table 4: Exact and Numerical solution using HBI for Example 2 .

x	Exact	HBI	Error
0.0	6.634320126E-16	6.634320126E-16	0.
0.2	0.5590169122545	0.5590169122547	2.21E-12
0.4	-0.9045084378512	-0.9045084378518	6.25E-12

0.6	0.9045084354472	0.9045084354474	2.57E-12
0.8	-0.5590169311454	-0.5590169311451	3.38E-12
1.0	-4.658833273E-16	-4.658833273E-16	0.

for $N = 10,\ x \in [-1,1]$.

Table 5: Comparison of maximum errors obtained in existing methods for Example 2 at $t = 1$

N	HBI l_∞ error	CPU Time	BVM l_∞ error	CPU Time	BUM l_∞ error	CPU Time
16	5.215E-8	0.214	9.662E-0	0.483	1.251E-1	0.531
32	2.125E-8	0.901	2.582E-2	1.235	2.578E-2	1.031
64	6.778E-8	3.114	6.433E-3	5.358	6.459E-3	5.516
128	8.974E-8	12.128	1.607E-3	43.641	1.607E-3	46.923
256	1.954E-8	32.125	2.000E-0	512.843	4.016E-4	532.657

BVM and BUM are Boundary Value Methods and the Block Unification Methods in [8].

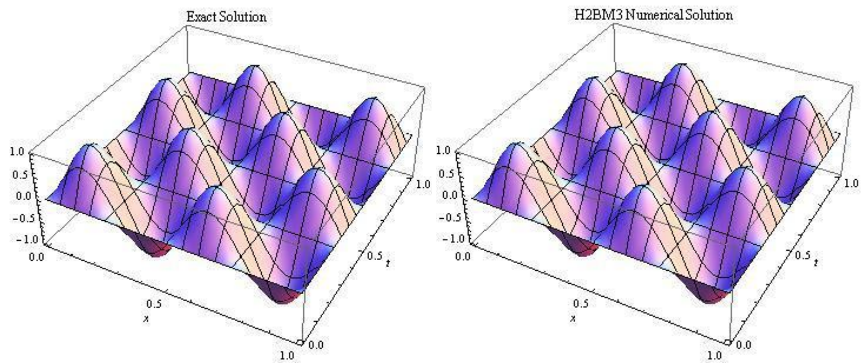

Figure 2: Surface plots for the Exact and numerical solution for Example 3

Table 4 shows the comparison of the exact and numerical solution and the errors for Example 2. Table 5 shows the maximum error and CPU time obtained for different methods. Comparing This show that the derived methods performs accurately, superiorly and affluently in terms of the computer time, and errors obtained for Examples 2. Figure 2 shows the surface plots for the exact and Numerical solution for Examples 2.

Example 3.

The temperature distribution of the radiation fin of trapezoidal is modeled to the following differential equation profile in [12].

$$\begin{cases} y'' - y - \sin(2\pi x)(-1-4\pi^2)\left(x^3 - \frac{4}{3}x^2 + \frac{x}{3}\right) \\ -\left(6x - \frac{8}{3}\right)\sin(2\pi x) - 4\pi\cos(2\pi x)\left(3x^2 - \frac{8}{3}x + \frac{1}{3}\right) = 0, \qquad 0 \le x \le 1 \\ y(0) = y(1), \qquad y'(0) = y'(1) \end{cases} \tag{32}$$

The analytic solution is given by $y(x) = \left(x^3 - \frac{4}{3}x^2 + \frac{x}{3}\right)\sin(2\pi x)$.

Table 6: Comparison of Maximum errors obtained in different methods for Example 3.

N	HBI	Method in [12]
16	2.154E-4	Nil
32	7.845E-6	5.1818E-3
64	3.789E-6	1.3008E-3
128	1.458E-6	3.2608E-4

Table 6 shows the comparison of Maximum errors obtained for Example 3 with the method in [12].

Example 4.

Consider the problem in [13].

$$\begin{cases} y'' - y = \cos(x), & 0 \le x \le 1 \\ y(0) = 0, \quad y(1) = 1 \end{cases} \tag{33}$$

The analytic solution is given by $y(x) = \cos(x) + \frac{1-\cos(1)}{\sin(1)}\sin(x) - 1$. With $N = 10$.

Table 7: Comparison of exact and numerical solutions obtained for Example 7.

X	Exact	HBI
0	0	0
0.1	0.059343034025940	0.059343033922545
0.2	0.110134207176555	0.110134206998541
0.3	0.151024408862577	0.151024408745541
0.4	0.180475345562389	0.180475345618845
0.5	0.196734670143683	0.196734670284321
0.6	0.197807972378616	0.197807973077854
0.7	0.181427245522797	0.181427246017854
0.8	0.145015397537614	0.145015399155465
0.9	0.085646323767636	0.085646324667561
1.0	0	0

Table 8: Comparison of errors obtained in different methods for Example 4.

x	Method in [14]	Method in [13]	HBI
0.1	1.130000E-07	1.980493E-14	4.12472E-16
0.2	2.190000E-07	3.832952E-14	8.88781E-16
0.3	3.290000E-07	5.551284E-14	3.21548E-16
0.4	3.740000E-07	6.681520E-14	8.47855E-16
0.5	4.170000E-07	7.463205E-14	3.54878E-16
0.6	4.680000E-07	7.869950E-14	3.21548E-16
0.7	4.280000E-07	7.251511E-14	3.21254E-16
0.8	3.620000E-07	5.918955E-14	4.77855E-16
0.9	2.620000E-07	3.811464E-15	1.47854E-16
1.0	N/A	0.0000000000	0.000000000

Table 7 shows clearly,the comparison of Exact and numerical solutions or Example 4. Table 8 shows comparison of errors obtained in different methods for Example 4. The Method HBI performed favourable well when compared to Methods in [13] and [14].

5.2 Conclusion

The development of some numerical schemes has been proposed in this work. This was developed via the interpolation and collocation techniques using power series function as trial solutions. The methods solves effectively second order partial differential equations (PDEs) and BVPs in Ordinary differential equations and the results obtained were accurate. The analysis of the new methods showed that all satisfy the properties of numerical methods for solution of differential equations. Namely, Consistency, Zero- Stability, Continuity and convergence.

References

[1] Lambert J. D., Computational methods in ordinary differential equation, John Wiley & Sons Inc. New York, USA. 1973

[2] L. Brugnano, D. Trigiante, "Convergence and stability of boundary value methods for ordinary differential equations", *Journal of Computational and Applied Mathematics*vol 66 pp. 97-109, (1996).

[3] Onumanyi,P;(1981); Numerical Solution of B.V.Ps with the Tau Method.*Phd Thesis University of London,England.*

[4] Fatunla S.O, (1999) Block Methods for second – order IVP,*Int .J.Mathematics* Vol.55

[5] Jator S. N. (2016) Trigonometric symmetric boundary value method for oscillating solutions including the sine-Gordon and Poisson equations *Cogent Mathematics 2016*, 3: 1271269

[6] Awoyemi D. O.,A new sixth order algorithms for General Second order ordinary differential equation. Inter. J. Computer Math., 2001;77:117 – 124.

[7] Yusuph, Y and Onumanyi, P,(1995); New multiple FDM through multistep collocation for y'' $f(x,y)$,*Proceedings of the conference, NMC Abuja*

[8] Biala T. A (2016) 1Computational Study of the Boundary Value Methods and the Block Unification Methods for $y'' = f(x, y, y')$, *Abstract and Applied Analysis 2016*, Article ID 8465103, 14 pages http://dx.doi.org/10.1155/2016/8465103

[9] Henrici, P., Discrete Variable Methods for ODEs, John Wiley, New York, USA. 1962.

[10] Modebei M. I., Adeniyi R. B., Jator S. N. and Ramos H. C. (2019) A block hybrid integrator for numerically solving fourth-order Initial Value Problems *Applied Mathematics and Computation 346* 680-694

[11] F. F. Ngwane and S. N. Jator, "Block hybrid method using trigonometric basis for initial value problems with oscillating solutions," *Numerical Algorithms*, vol. 63, no. 4, pp. 713–725, 2014.

[12] Siraj-ul-Islam, Imran Aziz and Bozidar Sarler (2010), The numerical solution of second-order boundary-value problems by collocation method with the Haar wavelets, *Mathematical and Computer Modelling 52* 1577-1590.

[13] O. Adeyeye and Z. Omar (2016), Maximal Order Block Method For The Solution Of Second Order Ordinary Differential Equations, *IAENG International Journal of Applied Mathematics, IJAM 46* (4) 22-28.

[14] Majid, Z. A., Nasir, N. M., Ismail, F. and Bachok, N. Two Point Diagonally Block Method for Solving Boundary Value Problems with Robin Boundary Conditions, Malaysian Journal of Mathematical Sciences 13(S) April: 1–14 (2019).