



Construction of Functions for Fractional Derivatives using Matlab

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Authors' contributions

This work was carried out in collaboration between both authors. Author OAS designed the study and wrote the first draft of the manuscript, author JFL carried out the MATLAB coding and edited the entire work. Both authors read and approved the final manuscript.

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Abstract

MATLAB is a high level programming tool for technical computing, its application cuts across different sphere of science, engineering, finance, communication, music etc. With the current increase in the use of non-integer order derivatives, there is a need to have tools that handle them for effective applications. In this paper, we present a brief comparative review of 2 expressions of fractional derivative. MATLAB functions for approximating Riemann-Liouville and Caputo fractional derivatives are presented alongside. Numerical simulations with test examples are implemented and results compared. To effectively handle non-polynomial function, Taylor series expansion is employed to convert the function into a form that can be easily handled.

Keywords: MATLAB; fractional derivative; caputo derivative; riemann liouville derivative.

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1 Introduction

Algorithms for effective evaluation of fractional derivative is herein discussed, an efficient and effective program for finding fractional derivative of any function via MATLAB. This task could be implemented in other computing environment like python, C, C++ etc but the programs contained in this paper were put together to exploit the strength of MATLAB as a language of technical computing [1] and a versatile problem solving environment (PSE) [2]. MATLAB is equipped with a host of inbuilt functions designed to effectively handle computations at different level, but function for handling fractional derivative is not yet incorporated into its library of functions. By fractional derivative, we mean the derivative of the form $D^\alpha f(x)$, where α is a non-integer order. The use of this derivative and its variant for partial derivative is ubiquitous in modern applications as they play significant role in analysing the behaviour of physical phenomena in different domains of science and engineering. These include viscoelasticity [3,4], biology [5], bioengineering [6], signal processing [7], diffusion processes [8], dynamical systems [9], allometry [10], heat and fluid flow [11]. These applications demand for evaluation of fractional derivatives contained in the equation. This evaluation goes with different kinds of definition for fractional derivative unlike classical derivative (integer order derivative) which has a single definition.

Fractional derivatives have gone through several phases of development over the years, these have resulted into a rich variety of fractional derivatives – from the discrete Grünwald-Letnikov fractional derivative defined in a coordinate space to a continuous Fourier fractional derivative defined in a frequency domain [12-14]. Also, there exists a good number of numerical approaches and analytical techniques that have been deployed in evaluating fractional derivatives for a wide spectrum of functions [13], most of these functions goes with individual limitations in handling some functions with relevant physical applications. For example, authors in [14] opined that method that led to exact evaluation of Caputo derivative of broad class of elementary functions was limited to a class of functions that can be expressed in terms of hypergeometric functions with a power-law argument. In their account, individual function that can be represented in terms of single hypergeometric function with a power-law argument are well taken care of, but their combination, in the most general case cannot be brought to such form, hence the limitation of the technique [14].

Among different kinds of fractional derivatives available in literatures, we in this study focus on the use of MATLAB functions in evaluating Riemann Liouville and Caputo fractional derivatives [15]. The choice of these duo is due to their fundamental role in applications and their wide acceptability among researchers [14]. We need to point out the fact that this form of derivative has been automated previously, however, practical implementation of fractional derivative as a function call in MATLAB that is capable of finding fractional derivative of complex functions is limited. In this work, we solve this problem by first finding Taylor series expansion of the function before applying fractional derivative on the expanded form in a much simpler and easy to automate approach.

2 Preliminary

Fractional derivative of functions $f(t)$ is the derivative of arbitrary real order α denoted

$$D_{a,t}^\alpha f(t)$$

Where $\alpha > 0$ is the order of the derivative $> a, \alpha, a, t \in \mathbb{R}$. The subscripts a and t are the two limits related to the operation of fractional differentiation and are referred to as terminals of the fractional differentiation, Ross [16]. The terminals are sometimes omitted for convenience. For the sake of ease of referencing three kinds of definition of fractional derivative are reviewed and compared, these are outlined as follow:

2.1 Riemann-Liouville fractional derivative

It has been observed that the use of some fractional derivative like Grünwald-Letnikov fractional derivatives is not convenient for non-integer terms [17]. The most widely known alternative is the Riemann-Liouville Definition given as the integro-differential expression.

$${}_{RL}D_{a,t}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \square \\ \frac{d^n}{dt^n} f(t) & \alpha = n \in \square \end{cases} \quad (3)$$

Some fractional derivatives go with an assumption that the function $f(t)$ must be $n + 1$ times continuously differentiable, except with very few exceptions. Riemann-Liouville's definition bypasses this condition on the function $f(t)$ as it only demands for integrability of $f(t)$. [17]

2.2 Caputo fractional derivatives

According to assertions contained in [4,17], mathematical modelling of a good number of physical phenomena such as viscoelasticity, solid mechanisms, biology etc demands for utilization of physically interpretable initial conditions such as the ones defined as $f(a), f'(a)$ etc, but the Riemann-Liouville Fractional definition leads to initial conditions containing the limit values of Riemann-Liouville fractional derivatives at the lower terminal $t = a$. [17,18]. For example

$$\begin{aligned} \lim_{t \rightarrow a} D_{a,t}^{\alpha-1} f(t) &= b_1 \\ \lim_{t \rightarrow a} D_{a,t}^{\alpha-2} f(t) &= b_2 \\ &\text{“ “} \\ &\text{“ “} \\ \lim_{t \rightarrow a} D_{a,t}^{\alpha-n} f(t) &= b_n \end{aligned}$$

Where $b_k, k = 1, 2, \dots, n$ are given constant. Regardless of the fact that initial value problems with such initial conditions can be successfully solved, their solutions are practically useless because there is no known physical interpretation for such type of initial conditions. To resolve this limitation, M. Caputo proposed fractional derivatives of the form

$${}_cD_{a,t}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & n-1 < \alpha < n \in \square \\ \frac{d^n}{dt^n} f(t) & \alpha = n \in \square \end{cases} \quad (4)$$

For $\alpha \rightarrow n$, the Caputo derivative resulted into integer-order n th derivative of the function $f(t)$. i.e.

$$\begin{aligned} \lim_{\alpha \rightarrow n} D^{\alpha} f(t) &= \lim_{\alpha \rightarrow n} \left(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_a^t f(t-\tau)^{n-\alpha} f^{(n+1)}(\tau) d\tau \right) \\ &= f^{(n)}(a) + \int_a^t f^{(n+1)}(\tau) d\tau \\ &= f^{(n)}(t); n = 1, 2, 3, \dots \end{aligned}$$

As in the case of the Riemann-Liouville derivative, Caputo's derivative equally provides an interpolation between integer-order derivatives [17]. However, the advantage of Caputo's definition is that the initial condition takes the same form as the integer-order differential equation i.e. It contains the limit values of integer-order derivatives of unknown functions at the lower terminal $t = a$.

In addition to this, Caputo derivative naturally converges at the origin. This property, in conjunction with its non-local nature makes it most suitable choice in a wide range of physical applications that involve fractional derivative [19-21].

3 Properties of Fractional Derivatives

There are several properties of fractional derivatives as given and proven in existing literatures [1], however, few ones that are relevant to this study are outlined in this section.

3.1 Linearity

Let $n - 1 < \alpha < n, n \in \mathbb{R}, \alpha, \lambda \in \mathbb{C}$ and the function $f(t)$ and $g(t)$ be such that a linear operator such that ${}_c D^\alpha f(t)$ and ${}_c D^\alpha g(t)$ exist. The Caputo fractional derivative ${}_c D^\alpha$ is a linear operator i. e ${}_c D^\alpha (\lambda f(t) + \mu g(t)) = \lambda {}_c D^\alpha f(t) + \mu {}_c D^\alpha g(t)$ similarly, the Riemann Liouville operator satisfies ${}_{RL} D^\alpha (\lambda f(t) + \mu g(t)) = \lambda {}_{RL} D^\alpha f(t) + \mu {}_{RL} D^\alpha g(t)$

3.2 Fractional derivatives of product of two functions

From Leibniz rule for evaluating nth derivative of the product of 2 functions $g(t) f(t)$ we have the fractional derivatives of product of two functions as:

$$D_{a,t}^\alpha (f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D_{a,t}^{\alpha-k} f(t)$$

Where $f(\tau)$ and $g(\tau)$ with their derivatives are continuous within $[a, t]$

3.3 Fractional derivatives of composite functions

$$D_{a,t}^\alpha F(\square(t)) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \varphi(t) + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{k! (t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^k F^{(m)}(\square(t)) \sum_{r=1}^k \frac{1}{a_r!} \left(\frac{\square^{(r)}(t)}{r!} \right)^{a_r}$$

where the summation extends over all combinations of all non-negative integer values of a_1, a_2, \dots, a_k such that

$$\sum_{r=1}^k r a_r = k \text{ and } \sum_{r=1}^k a_r = m.$$

3.4 Comparison of Riemann-Liouville and Caputo fractional derivatives

Riemann-Liouville and Caputo Fractional derivatives do not coincide. i.e. suppose $f(t)$ is a function of which both ${}_{RL} D^\alpha f(t)$ and ${}_c D^\alpha f(t)$ exist and $n - 1 < \alpha < n \in \mathbb{R}$ then in general

$${}_{RL} D^\alpha f(t) \neq {}_c D^\alpha f(t)$$

However, these derivatives only coincide if the function $f(t)$ be such that $f^{(s)} = 0, s = 0, 1, 2, \dots, n - 1$ then

$${}_{RL} D^\alpha f(t) = {}_c D^\alpha f(t)$$

The table given depicts the comparison between these two operators.

Table 1. Comparison between ${}_R D^\alpha$ and ${}_C D^\alpha$

Property	Riemann-Liouville ${}_R D^\alpha$	Caputo ${}_C D^\alpha$
Representation	$D^\alpha f(t) = D^n J^{n-\alpha} f(t)$	$D^\alpha f(t) = J^{n-\alpha} D^n f(t)$
Interpolation	$\lim_{\alpha \rightarrow n} D^\alpha f(t) = f^{(n)}(t)$ $\lim_{\alpha \rightarrow n-1} D^\alpha f(t) = f^{(n-1)}(t)$	$\lim_{\alpha \rightarrow n} D^\alpha f(t) = f^{(n)}(t)$ $\lim_{\alpha \rightarrow n-1} D^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$
Linearity	$D^\alpha(\lambda f(t) + g(t)) = \lambda D^\alpha f(t) + D^\alpha g(t)$	$D^\alpha(\lambda f(t) + g(t)) = \lambda D^\alpha f(t) + D^\alpha g(t)$
Non-commutation	$D^\alpha D^m f(t) = D^{\alpha+m} f(t) \neq D^m D^\alpha f(t)$	$D^\alpha D^m f(t) = D^{\alpha+m} f(t) \neq D^m D^\alpha f(t)$
Laplace Transform	$L\{D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0}$	$L\{D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$
Leibniz rule	$D^\alpha(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D^{\alpha-k} f(t)g^{(k)}(t))$	$D^\alpha(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D^{\alpha-k} f(t)g^{(k)}(t)) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} ((f(t)g(t))^{(k)}(0))$
$f(t) = c$ $= \text{constant}$	$D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha} \neq 0,$ $c = \text{constant} \neq 0$	$D^\alpha c = 0, \quad c = \text{const}$

4 MATLAB Function for Fractional Derivative

MATLAB (Matrix laboratory) is vastly endowed for technical computing, visualization and programming [2]. It has a host of in-built functions and also allows for user-defined function. A function in MATLAB is a collection of statements that work together to perform a computational task. They are defined in separate files whose name and that of the function itself must be the same. Every function (in-built and user-defined) operates on variables within their own workspace separate from the base workspace [1].

MATLAB is adequately equipped to evaluate classical derivatives of functions, either analytically or numerically. These are done via the following functions;

Table 2. Partial derivative is evaluated using the same procedure, but the variable with respect to which the function is to be differentiated must be specified

Operators	MATLAB Command
$\frac{df}{dx}$	diff(f) or diff (f,x)
$\frac{d^n f}{d^n x}$	diff(f, x, n)
$J = \frac{\partial(r, t)}{\partial(u, v)}$	J = Jacobian ([r; t] , [u; v])

diff command takes both function and polynomial as input and produces symbolic derivatives as output.

In this work, we build MATLAB function handle that uses the pre-discussed derivatives to determine fractional derivative $D_{a,t}^{\alpha} f(t)$ of a function. They are written as predefined function that can be called directly from the command window using function calls `difffr()` and `diffc()` for Riemann-Liouville and Caputo's fractional derivatives respectively. For non-polynomial functions, the programs first carry out Taylor series expansion via the function `taylor(f,4*m)`, where $m = \lceil \alpha \rceil$.

The syntax of a function statement is:

```
function[out1, out2, . . . , outN ]=myfun(in1, in2, . . . , inN)
```

a. MATLAB function for Riemann_Liouville derivatives

```
function dyR= difffr(y,alp)
syms x t
%function script that finds fractional derivative of function in
%variable x or t.
m=ceil(alp);

y_t=subs(y,x,t);
y_t2=taylor(y_t,);
QR=int(y_t2/((x-t)^(alp-m+1)),t,0,x);
dy_fracR=(gamma(m-alp))^(-1)*diff(QR,m);
dyR=vpa(dy_fracR);
end
```

b. MATLAB function for Caputo derivatives

```
function dyC= diffc(y,alp,)
syms x t
%function script that finds fractional derivative of function in
%variable x or t using Caputo's definition.
m=ceil(alp);

y_t=subs(y,x,t);
y_t2=taylor(y_t);
Qc=diff(y_t2,m)/((x-t)^alp-m+1);
dy_fracc=(gamma(m-alp))^(-1)*int(Qc,t,0,x);
dyc=vpa(dy_fracc);
dyc=expand(dyC)
end
```

5 Implementation

These functions are called directly from the command window to respectively implement Riemann-Liouville and Caputo derivatives as follows:

```
difffr(fun,alp)
diffc(fun,alp)
```

fun is the function whose fractional derivative is being sought for, it accepts alp as the order of fractional derivative. These effectively handle both polynomial and non-polynomial functions within a very minimal computational cost. These are implemented on function $f(t)$ with two fractional orders $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, results are compared across the two derivatives. The schemes worked effectively for both polynomial and non-polynomial functions, these are illustrated in the following table.

Table 3. These are illustrated in polynomial and non-polynomial functions

$f(x)$	Derivative of order $\alpha = \frac{1}{2}$	Derivative of order $\alpha = \frac{3}{2}$
$y = 5$	${}_R D = 2.8209x^{\frac{1}{2}}$ ${}_C D = 0$	${}_R D = -1.4104x^{\frac{3}{2}}$ ${}_C D = 0$
$y = x^2$	${}_R D = {}_C D = 1.5045x^{\frac{3}{2}}$	${}_R D = {}_C D = 2.2568x^{\frac{1}{2}}$
$y = \sin x$	${}_R D = {}_C D = 1.1284x^{\frac{1}{2}} - 0.309x^{\frac{5}{2}}$	${}_R D = {}_C D = -0.7523x^{\frac{3}{2}}$
$y = \cos x$	${}_R D = {}_C D = 0.0860x^{\frac{7}{2}} - 0.7523x^{\frac{3}{2}}$	${}_R D = {}_C D = 0.3009x^{\frac{5}{2}} - 1.1284x^{\frac{1}{2}}$
$y = 5x^2 - 4x + 6$	${}_R D = 3.3851x^{\frac{1}{2}} - 4.5135x^{\frac{1}{2}}$ $+ 7.5225x^{\frac{3}{2}}$ ${}_C D = 7.5225x^{\frac{3}{2}} - 4.5135x^{\frac{1}{2}}$	${}_R D = 11.2838x^{\frac{1}{2}} - 2.2568x^{\frac{1}{2}}$ $- 1.6926x^{\frac{3}{2}}$ ${}_C D = 11.2838x^{\frac{1}{2}} - 2.2568x^{\frac{1}{2}}$

Graphs of Fractional derivatives for some functions for different fractional order derivatives

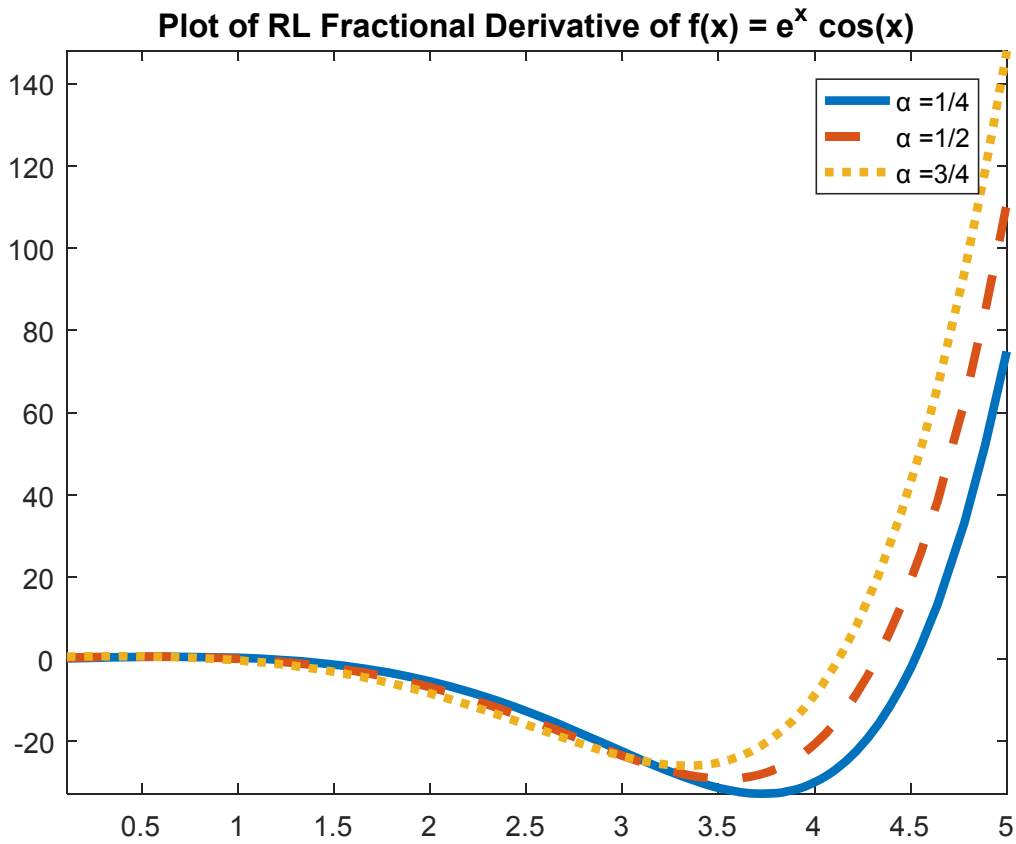


Fig. 1. Plot of R-L fractional derivative of $f(x) = e^x \cos x$

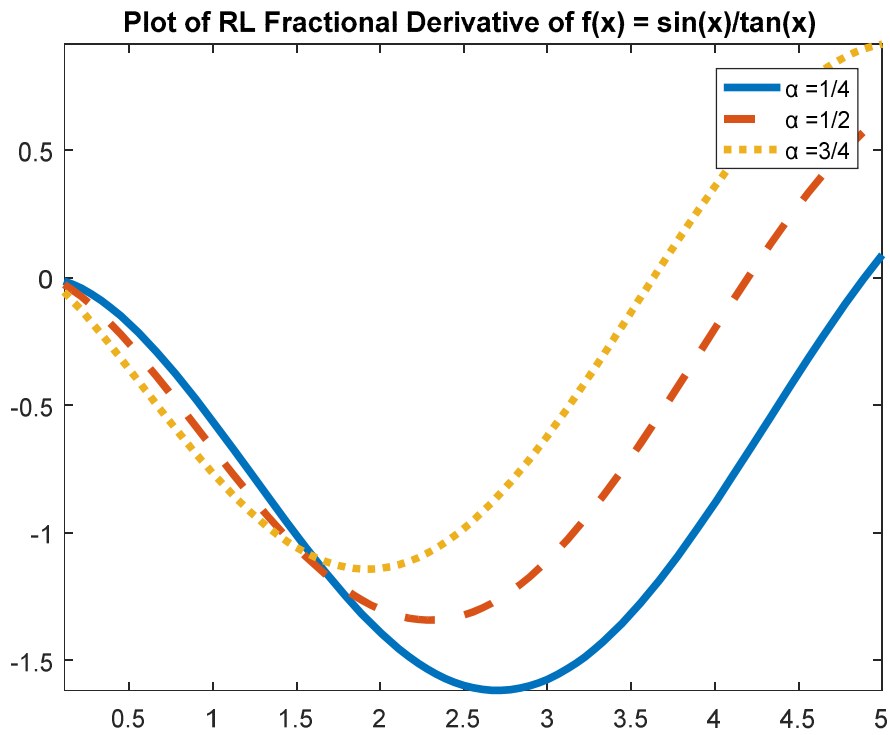


Fig. 2. Plot of RL fractional derivative of $f(x) = \sin x / \tan x$

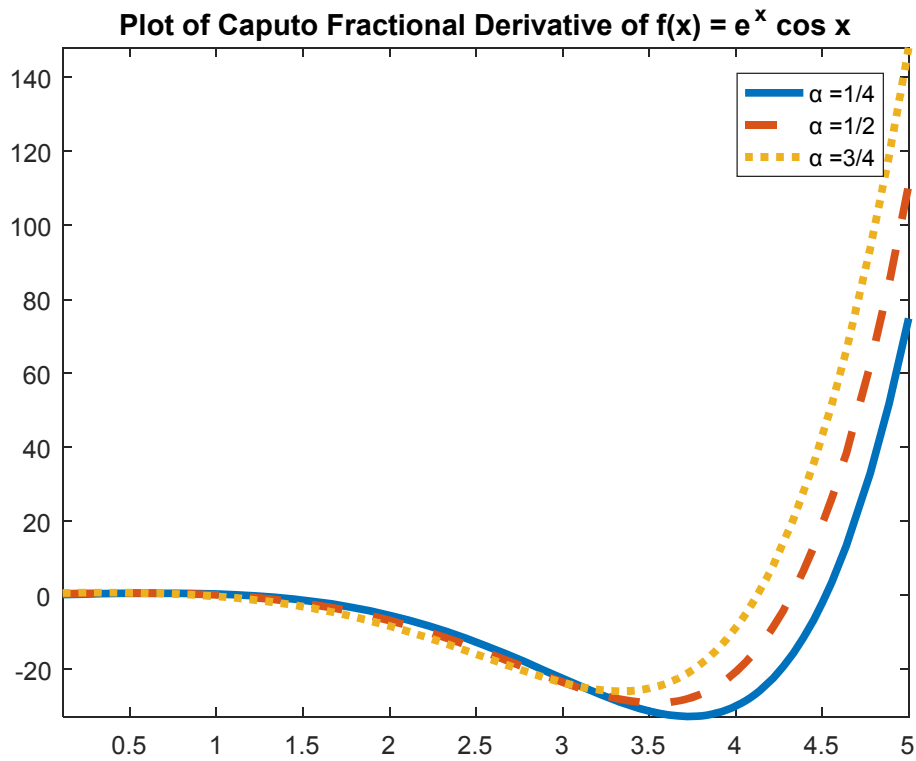


Fig. 3. Plot of Caputo fractional derivative of $f(x) = e^x \cos x$

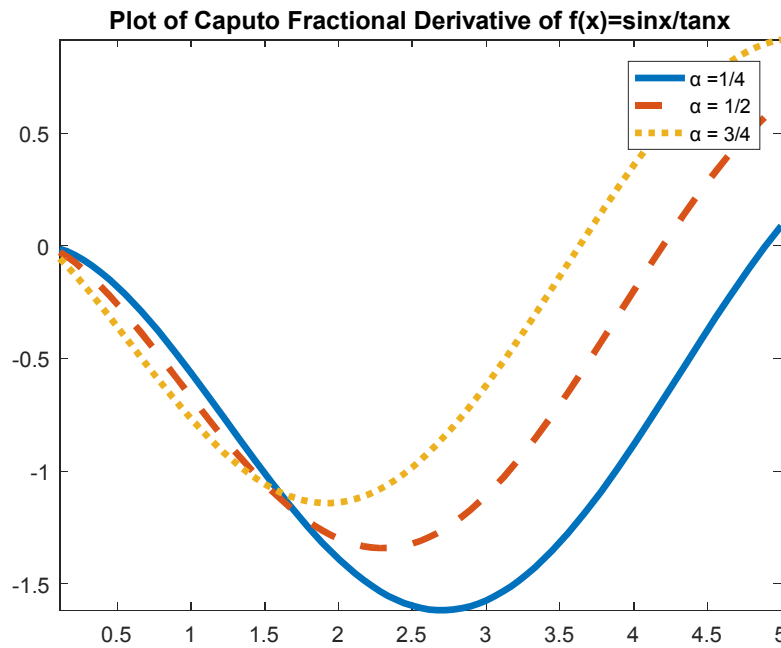


Fig. 4. Plot of Caputo fractional derivative of $f(x) = e^x \cos x$

6 Conclusion

A comparative review of both Riemann Liouville and Caputo fractional derivatives has been carried out. Fractional derivatives proposed by Riemann-Liouville and Caputo are used in constructing MATLAB function for fractional derivatives. Examples are given to illustrate the simplicity and applicability of the MATLAB functions. This study could be extended to cover fractional derivatives of special functions.

Competing Interests

Authors have declared that no competing interests exist.

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