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Linearization of the Equation of Motion of a Free Particle in a Space of Constant Curvature through Differential Forms

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Abstract
In this paper, we obtain the linearizing point transformation for the equation of motion of a free particle in a space of constant curvature using the method of differential forms.

Key Words: *Point transformation, Differential forms, Linearization, Free particle, Space of constant curvature,Second order ordinary differential equations.*

1. Introduction
The equation of motion of a free particle in a space of constant curvature is a second order ordinary differential equation. It was considered by some authors like [4] and [5] using Symmetry Group Classification (SGC) method and the Generalized Sundman Transformations (GST) method respectively.

Linearization method using differential forms was derived by [3] for second order ordinary differential equations. The method provided a simple understanding of the linearization problem. It is important to state that, linearization method in general, has to do with point transformation (PT) and non-point transformation (NPT) [1]. Point transformation preserves the integrability of the equation and its Lie symmetry structure [2], and hence the reason for the use of point transformation.

In this paper, we construct the linearizing point transformation for the equation of motion of a free particle in a space of constant curvature using the method of differential forms derived by [3].

2. The Method
Our starting point is a second order ordinary differential equation

$$y'' = f(x, y, y'). \qquad (2.1)$$

We assume a point transformation given by the variables

$$X = F(x, y), \quad Y = G(x, y), \qquad (2.2)$$

with a requirement that,

$$\frac{d^2Y}{dX^2} = 0. \qquad (2.3)$$

We first construct, using equation (2.2)

$$\frac{dY}{dX} = \frac{G_x + G_y y'}{F_x + F_y y'} \qquad (2.4)$$

where $F_x + F_y y' \neq 0$ and the subscripts x and y denote partial differentiation. The second derivative equation may be written simply in terms of a differential $d\left(\frac{dY}{dX}\right) = 0$ which becomes

$$(F_x + F_y y')(dG_x + y' dG_y + G_y dy') -$$

$\left(G_x + G_y y'\right)\left(dF_x + y'dF_y + F_y dy'\right) = 0.$ (2.5)

We can expand (2.5) and write it as

$$T dy' + \rho y'^2 + (\lambda + \delta) y' + \sigma = 0, \quad (2.6)$$

where

$$T = F_x G_y - F_y G_x, \quad (2.7)$$

and we have the 1-forms

$$\left.\begin{aligned} \rho = F_y dG_y - G_y dF_y, \lambda = F_y dG_x - G_y dF_x, \\ \sigma = F_x dG_x - G_x dF_x, \delta = F_x dG_y - G_x dF_y. \end{aligned}\right\} \quad (2.8)$$

We can rewrite equation (2.6) as

$$dy' = \alpha + \beta y' + \gamma y'^2, \quad (2.9)$$

where

$$\alpha = \frac{-\sigma}{T}, \beta = \frac{-(\lambda+\delta)}{T}, \gamma = \frac{-\rho}{T}. \quad (2.10)$$

For integrability of equation (2.9) we set $ddy' = 0$, that is

$$0 = d\alpha + dy' \wedge \beta + y' d\beta + 2y' dy' \wedge \gamma + y'^2 d\gamma. \quad (2.11)$$

Substituting (2.9) into equation (2.11), we have:

$$0 = d\alpha + (\alpha + \beta y' + \gamma y'^2) \wedge \beta + y' d\beta + 2y'(\alpha + \beta y' + \gamma y'^2) \wedge \gamma + y'^2 d\gamma.$$
(2.12)

The y'^3term in equation (2.12) vanishes because $\gamma \wedge \gamma = 0$, we expand equation (2.12) and equate the coefficients of the other powers of y' to zero to have:

$$d\alpha = \beta \wedge \alpha, d\beta = 2\gamma \wedge \alpha, dr = \gamma \wedge \beta. \quad (2.13)$$

Now, we go back to equations (2.8) and expand the differentials, to have:

$$\rho = F_y\left(G_{xy} dx + G_{yy} dy\right) - G_y\left(F_{xy} dx + F_{yy} dy\right),$$
$$\lambda = F_y(G_{xx} dx + G_{xy} dy - G_y\left(F_{xx} dx + F_{xy} dy\right),$$
$$\sigma = F_x\left(G_{xx} dx + G_{xy} dy\right) - G_x\left(F_{xx} dx + F_{xy} dy\right),$$
$$\delta = F_x\left(G_{xy} dx + G_{yy} dy\right) - G_x\left(F_{xy} dx + F_{yy} dy\right),$$

which can simply be written as

$$\rho = Adx + Bdy, \lambda = Cdx + Ady, \sigma = Ddx + Edy, \delta = Edx + Hdy, \quad (2.14)$$

where

$$A = F_y G_{xy} - G_y F_{xy}, B = F_y G_{yy} - G_y F_{yy}$$
$$C = F_y G_{xx} - G_y F_{xx}, D = F_x G_{xx} - G_x F_{xx}$$
$$E = F_x G_{xy} - G_x F_{xy}, H = F_x G_{yy} - G_x F_{yy}.$$

Thus,

$$\alpha = \frac{-(Ddx+Edy)}{T}, \beta = \frac{-(Cdx+Edx+Ady+Hdy)}{T}, \gamma = \frac{-(Adx+Bdy)}{T}. \quad (2.15)$$

Substituting α, β and γ into equation (2.9) and dividing by dx to convert the differential forms to functions, we have:

$$y'' + f_0 + f_1 y' + f_2 y'^2 + f_3 y'^3 = 0, \quad (2.16)$$

where the f_k are given by

$$f_0 = \frac{D}{T}, f_1 = \frac{(C+2E)}{T}, f_2 = \frac{(H+2A)}{T}, f_3 = \frac{B}{T}. \quad (2.17)$$

We define K and L as

$$K = \frac{E}{T}, L = \frac{A}{T}, \quad (2.18)$$

and replace D, C, H and B in the 1-forms in equation (2.15) in favour of the f_k, K and L, obtaining

$$\alpha = -f_0 dx - Kdy, \beta = (K - f_1)dx + (L - f_2)dy, \gamma = -Ldx - f_3 dy. \quad (2.19)$$

We also note that

$$\frac{dT}{T} = (3K - f_1)dx + (f_2 - 3L)dy. \qquad (2.20)$$

We see that the 1-forms α, β, γ in (2.19)and $\frac{dT}{T}$in equation (2.20) are now expressed in terms of these four known functions K and L. The first three of these 1-forms can now be substituted into equation (2.13) on the various functions. If we do that, the first equation for $d\alpha$, gives the equation

$$f_{0y} - K_x = -K(K - f_1) + f_0(L - f_2) \qquad (2.21)$$

which is nonlinear in K. The other equations give the results:

$$-K_y + f_{1y} + L_x - f_{2x} = 2KL - f_0 f_3 \qquad (2.22)$$

and

$$L_y - f_{3x} = -L(L - f_2) + f_3(K - f_1) \qquad (2.23)$$

which are also nonlinear. However, we can simplify the situation by defining new variables:

$$T = \frac{1}{W^3},\ E = \frac{U}{W^4},\ A = \frac{V}{W^4}, \qquad (2.24)$$

So that from (2.18)

$$K = \frac{U}{W},\ L = \frac{V}{W}, \qquad (2.25)$$

and from (2.20)

$$3\frac{dW}{W} = (f_1 - 3K)dx + (3L - f_2)dy. \qquad (2.26)$$

We now have this situation. The dWequation (2.26) gives expressions for W_x and W_y. The equation (2.21) gives, after substitution for W_x, an expression

$$U_x = Wf_{0y} - \frac{2}{3}Uf_1 - Vf_0 + Wf_0f_2 \qquad (2.27)$$

which is linear in $U,\ V$and W. The equation (2.23) gives an expression

$$V_y = Wf_{3x} + \frac{2}{3}Vf_2 + Uf_3 - Wf_1f_3 \qquad (2.28)$$

which is also linear. The equation (2.22) gives a linear expression

$$V_x - U_y = \frac{U}{3}f_2 + \frac{V}{3}f_1 - Wf_{1y} + Wf_{2x} - 2f_0f_3W. \qquad (2.29)$$

The integrability condition on (2.26) gives a linear expression

$$V_x + U_y = \frac{U}{3}f_2 + \frac{V}{3}f_1 + \frac{W}{3}f_{2x} + \frac{W}{3}f_{1y}. \qquad (2.30)$$

Equations (2.29) and (2.30) can be solved for $V_x$and U_y. Thus we have expressions for all derivatives of $U,\ V$ and W, all of which are linear and homogeneous in the same variables. That is

$$dU = \frac{1}{3}\Big(-2Uf_1 - 3Vf_0 + W\big(3f_{0y} + 3f_0f_2\big)\Big)dx + \frac{1}{3}\Big(-Uf_2 + W\big(2f_{1y} - f_{2x} + 3f_0f_3\big)\Big)dy, \qquad (2.31)$$

$$dV = \frac{1}{3}\Big(Vf_1 + W\big(2f_{2x} - f_{1y} - 3f_0f_3\big)\Big)dx + \frac{1}{3}\big(3Uf_3 + 2Vf_2 + W(3f_{3x} - 3f_1f_3)\big)dy, \qquad (2.32)$$

$$dW = \frac{1}{3}(-3U + Wf_1)dx + \frac{1}{3}(3V - Wf_2)dy. \qquad (2.33)$$

We summarize all these relations in a nice matrix equation

$$dr = Mr, \qquad (2.34)$$

where

$$r = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \text{and } M = Pdx + Qdy,$$

$$P = \left(\frac{1}{3}\right)\begin{pmatrix} -2f_1 & -3f_0 & 3f_{0y} + 3f_0 f_2 \\ 0 & f_1 & 2f_{2x} - f_{1y} - 3f_0 f_3 \\ -3 & 0 & f_1 \end{pmatrix}$$

$$Q = \left(\frac{1}{3}\right)\begin{pmatrix} -f_2 & 0 & 2f_{1y} - f_{2x} + 3f_0 f_3 \\ 3f_3 & 2f_2 & 3f_{3x} - 3f_1 f_3 \\ 0 & 3 & -f_2 \end{pmatrix}.$$

For integrability of (2.34), $ddr = 0$ giving

$$dM = M \wedge M \qquad (2.35)$$

which is not zero since M is a matrix. Substitution for M in terms of P and Q gives the condition

$$Q_x - P_y + QP - PQ = 0. \qquad (2.36)$$

This matrix condition in (2.36) reduces to two equations:

$$f_{0yy} + f_0\left(f_{2y} - 2f_{3x}\right) + f_2 f_{0y} - f_3 f_{0x} + \left(\tfrac{1}{3}\right)\left(f_{2xx} - 2f_{xy} + f_1 f_{2x} - 2f_1 f_{1y}\right) \qquad (2.37)$$

and

$$f_{3xx} + f_3\left(2f_{0y} - f_{1x}\right) + f_0 f_{3y} - f_1 f_{3x} + \left(\tfrac{1}{3}\right)\left(f_{1yy} - 2f_{2xy} + 2f_2 f_{2x} - f_2 f_{1y}\right) = 0. \qquad (2.38)$$

To summarize, we note that the original differential equation is cubic in y', with the coefficients satisfying equations (2.37) and (2.38).

Now, we shall construct the point transformations proper. We will need U, V and W therefore we need to solve equations (2.34). Once the equations are solved, we construct K and L from equation (2.25).

In order to find the $F(x, y)$ and $G(x, y)$ for which we are seeking, we revert to equations (2.8) and solve for dF_x, dF_y, dG_x and dG_x. Solution for dF_x and dF_y gives

$$dF_x = \frac{\left(F_y\sigma - F_x\lambda\right)}{T}, \qquad dF_y = \frac{\left(F_y\delta - F_x\rho\right)}{T}.$$

Solution for dG_x and dG_y, shows that they satisfy the same equation, so we will write only equations for the derivatives of F. We note that

$$\delta + \lambda = -T\beta \text{ and } \delta - \lambda = dT,$$

so we can solve these equations for δ and λ. We can also substitute for σ and ρ in terms of α and γ. We get finally

$$dF_x = -F_y\alpha + F_x\frac{\left(\beta + \frac{dT}{T}\right)}{2}, \qquad dF_y = F_x\gamma + F_y\frac{\left(-\beta + \frac{dT}{T}\right)}{2}.$$

We substitute for α, β, γ and dT/T from equations (2.19) and (2.20) respectively in terms of the expressions obtained above, with the f_k, K and L.

We now have two equations which can be expressed in matrix form as follows;

$$dR = ZR, \; dS = ZS \qquad (2.39)$$

where

$$Z = \begin{pmatrix} (2K - f_1)dx - Ldy & f_0 dx + Kdy \\ -Ldx - f_3 dy & Kdx + (f_2 - 2L)dy \end{pmatrix},$$

$$R = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} G_x \\ G_y \end{pmatrix}.$$

This linear equation set can be solved for R. There will be two independent solutions, which can be taken as R and S as seen in equation (2.39). Integrability is guaranteed by setting $ddR = 0$. One can solve:

$$dF = (dx \quad dy)R$$

$$dG = (dx \quad dy)S \quad (2.40)$$

for F and G.

We can summarize the procedure as follows:

1. Make sure that the original differential equation is a cubic in y'as in equation (2.16)
2. Test the coefficients $f_k$to see whether they satisfy equations (2.37) and (2.38).
3. Construct the 3×3 matrix M and solve equation (2.34) (linear) for the three components of $r -$ a special solution is usually sufficient and construct K and$_L$.
4. Construct the 2×2 matrix Z and solve equation (2.39) (linear) for R or S.
5. Solve equation (2.40); the two independent solutions may be taken as F and G.

3. Construction of the Point Transformation

The equation of motion of a free particle in a space of constant curvature given by

$$y'' + 3yy' + y^3 = 0 \qquad (3.1)$$

was also considered by [4], using the method of symmetry group classification of ordinary differential equations: survey of some results.

The equation has the coefficients:

$$f_0 = y^3, f_1 = 3y,$$

$$f_2 = f_3 = 0$$

which satisfied the linearizability conditions in equations (2.37) and (2.38). Construction of 3×3 matrix

$M = Pdx + Qdy$ we have; $M = \begin{pmatrix} -2ydx & -y^3dx & 3y^2dx + 2dy \\ 0 & ydx & -dx \\ -dx & dy & ydx \end{pmatrix}$

so that $dr = \begin{pmatrix} -2yUdx - y^3Vdx + W(3y^2dx + 2dy) \\ yVdx - Wdx \\ -Udx + Vdy + yWdx \end{pmatrix}$ where $r = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$ and $dr = Mr$.

We let $U = 0, dU = 0$ and $dV = yVdx - Wdx$, $dW = Vdy + yWdx$. We can see that $W_x = yW$ and $W_y = V$. On integration, we obtain $W = e^{a(y)+xy}$ for some function a(y). But $V = W_y$, therefore, $V = e^{a(y)+xy}(x + a'(y))$. We use a special solution $a(y) = 1$ so that $U = 0$, $V = xe^{1+xy}$ and $W = e^{1+xy}$ so that $K = \frac{U}{W} = 0$, $L = \frac{V}{W} = x$.

Next we construct the 2 by 2 matrix Z which is $Z = \begin{pmatrix} -3ydx - xdy & y^3dx \\ -xdx & -2xdy \end{pmatrix}$.

Setting $R = \begin{pmatrix} b \\ c \end{pmatrix}$, we see that $dR = \begin{pmatrix} -b(3ydx + xdy) + cy^3dx \\ -bxdx - 2cxdy \end{pmatrix}$

so that $db = (-3by + cy^3)dx - bxdy$ and $dc = -bxdx - 2cxdy$.

Considering $b_y = -bx$, we have

$$b = ke^{-xy}, \qquad (3.2)$$

where k is a constant.

Differentiating equation (3.2) with respect to y, we see that $b_y = -kxe^{-xy}$. Also, $c_x = -bx$, that is $c_x = -kxe^{-xy}$. This is obvious that $c_x = b_y$. Integrating, we have as follows

$$c = -k\int xe^{-xy}dx + g(y)$$

$$c = -k\left[-\frac{x}{y}e^{-xy} + \frac{1}{y}\int e^{-xy}dx\right] + g(y)$$

$$c = kxy^{-1}e^{-xy} + ky^{-2}e^{-xy} + g(y) \qquad (3.3)$$

Differentiating equation (3.3) with respect to y we have:

$$c_y = \frac{-kxe^{-xy}}{y^2}(xy+1) - \frac{ke^{-xy}}{y^3}(xy+2) + g'(y) \qquad (3.4)$$

We also note that

$$c_y = -2cx. \qquad (3.5)$$

Equating equations (3.4) and (3.5) and simplifying we have:

$$g' + 2xg = 2ky^{-3}e^{-xy} - kx^2y^{-1}e^{-xy} \qquad (3.6)$$

Using the integrating factor with $P = 2x,\ Q = 2ky^{-3}e^{-xy} - kx^2y^{-1}e^{-xy}$, we see that: I.F= $e^{\int Pdy} = e^{\int 2xdy} = e^{2xy}$. Therefore $g \times I.F = \int(Q \times I.F)dy + m$becomes

$$ge^{2xy} = \int(2ky^{-3}e^{-xy} - kx^2y^{-1}e^{-xy})e^{2xy}dy + m,$$

where m is another constant apart from k. Integrating the above and simplifying, we have that

$$g = -ky^{-2}e^{-xy} - kxy^{-1}e^{-xy} + me^{-2xy}. \qquad (3.7)$$

Therefore equation (3.3) becomes

$$c = kxy^{-1}e^{-xy} + ky^{-2}e^{-xy} - ky^{-2}e^{-xy} - kxy^{-1}e^{-xy} + me^{-2xy},$$

which is reduced to

$$c = me^{-2xy} \qquad (3.8)$$

Summarizing, $b = ke^{-xy}$ and $c = me^{-2xy}$. Now, if $dF = bdx + cdy$, then, $b = F_x$and $c = F_y$.

Considering $F_y = me^{-2xy}$, on integration, we obtain:

$$F = \frac{-me^{-2xy}}{2x} + h(x). \qquad (3.9)$$

Now differentiating equation (3.9) with respect to x we see that

$$F_x = \frac{mye^{-2xy}}{x} + \frac{m}{2x^2}e^{-2xy} + h'(x).$$

Therefore, $mx^{-1}ye^{-2xy} + \frac{m}{2x^2}e^{-2xy} + h'(x) = ke^{-xy}$ or simply

$$h'(x) = ke^{-xy} - mx^{-1}ye^{-2xy} - \frac{m}{2x^2}e^{-2xy}. \qquad (3.10)$$

On integration of equation (3.10) by parts after truncating the last term since it is also the coefficient of the constant m we have: $h(x) = \frac{-k}{y} e^{-xy} - mylnxe^{-2xy}$.

Therefore, equation (3.9) becomes $F = \frac{-k}{y} e^{-xy} - \frac{m}{2x} e^{-2xy} - \; mylnxe^{-2xy}$ or

$$F + ke^{-xy}\left(\frac{1}{y}\right) + \; me^{-2xy}\left(\frac{1}{2x} + ylnx\right).$$

Without loss of generality, we let $e^{-xy} = e^{-2xy} = lnx = 1$, and interchanging the coefficient of m we have $X = \frac{1}{y}, \quad Y = \frac{1}{y} + x$ as the linearizing point transformation.

4. Conclusion

The equation of motion of a free particle in a space of constant curvature given in equation (3.1) was considered by two authors using two different methods: the (SGS) and the (GST). Their methods however, pose a difficulty in understanding the linearization problem. It is on this note that, we considered the same equation using the method of differential forms to give a clear understanding of the linearization problem.

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